Exact Design of Stepped-Impedance Transformers

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***Abstract*—The exact solutions of the design of stepped-impedance transformers presented until now have been tedious and subject to computational error. We present here an exact synthesis procedure which requires less effort than other exact procedures. In addition, two recurrence formulas are given for determining the characteristic impedances of each section. The coefficients so obtained can be compared to eliminate computational errors.**

Stepped-impedance transformers (SIT's) have been discussed frequently in the literature. Exact [1]–[3], approximate [4], and graphical solutions [5], as well as design tables [3], [6], have been published. The exact solutions published thus far are tedious and susceptible to computational error. In this short paper we present an exact synthesis procedure which can be carried out with less effort than is necessary with some other procedures.

The SIT consists of n lossless transmission-line sections, each of electrical length ϕ, terminated in resistive loads R_g and R_L. The input impedance of the r section, looking to the left, is

$$Z_{\text{in},r}(\phi) = Z_r \frac{Z_{\text{in},r-1}(\phi) + jZ_r \tan\phi}{Z_r + jZ_{\text{in},r-1}(\phi)\tan\phi} \tag{1}$$

where $Z_{\text{in},r-1}(\phi)$ is the input impedance of the $(r-1)$ section and Z_r is the characteristic impedance of the r section.

If we introduce the frequency variable $s=j\tan\phi$ [1] and define

$$z_r(s) = \frac{Z_{\text{in},r}(s)}{Z_{r+1}} \tag{2}$$

$$\zeta_r = \frac{Z_r}{Z_{r+1}} \tag{3}$$

we may write a normalized form of (1) as

$$z_r(s) = \zeta_r \frac{z_{r-1}(s) + s}{1 + z_{r-1}(s)s} \tag{4}$$

from which we note that

$$\zeta_r = z_r(1) = -z_r(-1). \tag{5}$$

In these equations, $r=1, 2, \cdots, n$, $Z_{n+1}=R_L$, $Z_0=R_g$, and $R=R_L/R_g=Z_{n+1}/Z_0$. We may solve (4) to obtain

$$z_{r-1}(s) = \frac{\zeta_r s - z_r(s)}{z_r(s)s - \zeta_r}. \tag{6}$$

If we know $z_n(s)$, we see from (5) and (6) that we may find $z_r(s)$ and ζ_r for all r. Further, from a knowledge of $Z_{n+1}(=R_L)$, (3) may be used successively to find the characteristic impedances Z_r of all sections of the SIT.

The direct use of (6) to determine the normalized impedance functions is not convenient, and we will develop a more convenient method. For a lossless SIT we may express $z_r(s)$ as

$$z_r(s) = \sum_{k=0}^{r} N_k{}^r s^k \Big/ \sum_{k=0}^{r} M_k{}^r s^k. \tag{7}$$

We substitute (7) into (6), which becomes

$$z_{r-1}(s) = \sum_{k=0}^{r+1} (\zeta_r M_{k-1}{}^r - N_k{}^r)s^k \Big/ \sum_{k=0}^{r+1} (N_{k-1} - \zeta_r M_k{}^r)s^k \tag{8}$$

if we note that $M_{-1}{}^r = N_{-1}{}^r = M_{r+1}{}^r = N_{r+1}{}^r = 0$.

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From (8), $z_{r-1}(s)$ appears at first to contain polynomials of higher order than $z_r(s)$. However, if we substitute $s=\pm 1$ into (6) and use (5), we see that both numerator and denominator are zero. Thus the factor (s^2-1) may be removed from both numerator and denominator of $z_{r-1}(s)$.

Let us write $z_{r-1}(s)$ as

$$z_{r-1}(s) = \sum_{k=0}^{r-1} N_k{}^{r-1} s^k \Big/ \sum_{k=0}^{r-1} M_k{}^{r-1} s^k \tag{9}$$

where we wish to determine the coefficients $M_k{}^{r-1}$ and $N_k{}^{r-1}$ from the known coefficients $M_k{}^r$ and $N_k{}^r$. We multiply the numerator and denominator of (9) by (s^2-1) and compare the result to (8), and obtain

$$N_{k-2}{}^{r-1} - N_k{}^{r-1} = \zeta_r M_{k-1}{}^r - N_k{}^r \tag{10a}$$

$$M_{k-2}{}^{r-1} - M_k{}^{r-1} = N_{k-1}{}^r - \zeta_r M_k{}^r. \tag{10b}$$

These equations may be solved to give

$$N_k{}^{r-1} = (N_k{}^r + N_{k-2}{}^r + N_{k-4}{}^r + \cdots) - \zeta_r(M_{k-1}{}^r + M_{k-3}{}^r + M_{k-5}{}^r + \cdots) \tag{11a}$$

$$M_k{}^{r-1} = \zeta_r(M_k{}^r + M_{k-2}{}^r + M_{k-4}{}^r + \cdots) - (N_{k-1}{}^r + N_{k-3}{}^r + N_{k-5}{}^r + \cdots) \tag{11b}$$

where the series are continued over positive subscripts only, in accord with our statement following (8). We may also obtain from (10) a different form,

$$N_k{}^{r-1} = \zeta_r(M_{k+1}{}^r + M_{k+3}{}^r + M_{k+5}{}^r + \cdots) - (N_{k+2}{}^r + N_{k+4}{}^r + N_{k+6}{}^r + \cdots) \tag{12a}$$

$$M_k{}^{r-1} = (N_{k+1}{}^r + N_{k+3}{}^r + N_{k+5}{}^r + \cdots) - \zeta_r(M_{k+2}{}^r + M_{k+4}{}^r + M_{k+6}{}^r + \cdots) \tag{12b}$$

with the series continued only for subscripts less than or equal to superscripts.

From (11) we may obtain the compact recurrence relations

$$N_k{}^{r-1} = N_k{}^r - M_{k-1}{}^{r-1} \tag{13a}$$

$$M_k{}^{r-1} = \zeta_r M_k{}^r - N_{k-1}{}^{r-1} \tag{13b}$$

and from (12) we get

$$N_k{}^{r-1} = \zeta_r M_{k+1}{}^r - M_{k+1}{}^{r-1} \tag{14a}$$

$$M_k{}^{r-1} = N_{k+1}{}^r - N_{k+1}{}^{r-1}. \tag{14b}$$

Equations (13) or (14)—or alternatively (11) or (12)—may be used to reduce the order of the normalized impedance function each time they are applied. Thus, by their use, if $z_n(s)$ is given, all of the n characteristic impedances of the SIT may be found. Note that if (11) or (13) is used, we find a coefficient in terms of the coefficients of equal and lower powers of s, whereas, (12) or (14) gives a coefficient in terms of the coefficients of equal and higher powers of s. This gives us a highly useful check on computational errors in determining the coefficients.

In general the power reflection coefficient for a lossless n-section SIT may be written as [4]

$$|\rho(\phi)|^2 = \frac{L_{2n}(\cos\phi)}{1 + L_{2n}(\cos\phi)} \tag{15}$$

where $L_{2n}(\cos\phi)$ is an even polynomial of degree $2n$. By the transformation $s=j\tan\phi$ we may write the voltage reflection coefficient and the input impedance function as

$$\rho(s) = \frac{Q_n(s)}{P_n(s)} \tag{16}$$

$$z_n(s) = \frac{P_n(s) + Q_n(s)}{P_n(s) - Q_n(s)} \tag{17}$$

where $Q_n(s)$ and $P_n(s)$ are positive real polynomials of degree n. For passive networks, $\rho(s)$ may be found uniquely from $|\rho(s)|^2$ [7], [8].

For a Chebyshev transformer, we know that

$$|\rho(\phi)|^2 = \frac{w^2 T_n^2\left(\frac{\cos\phi}{\cos\phi_l}\right)}{1 + w^2 T_n^2\left(\frac{\cos\phi}{\cos\phi_l}\right)} \qquad w = \frac{|\sqrt{R} - 1/\sqrt{R}|}{2T_n(\sec\phi_l)} \tag{18}$$

where T_n is the Chebyshev polynomial of degree n, and ϕ_l is defined as $\phi_l = \pi/(1+f_2/f_1)$, with f_1 and f_2 the lower and upper passband frequencies.

We write $\rho(s)$ in the form

$$\rho(s) = \frac{1-R}{1+R}\,\frac{\prod_{k=1}^{n}\left(1 - \frac{s}{q_k}\right)}{\prod_{k=1}^{n}\left(1 - \frac{s}{p_k}\right)} \tag{19}$$

where p_k and q_k are the poles and zeros of $|\rho(s)|^2$. By change of variable and letting

$$H = 2n\cosh^{-1}(\sec\phi_l) \tag{20}$$

$$W = 2\sinh^{-1}(1/w) \tag{21}$$

we obtain

$$p_k^2 = 1 - \frac{\cosh^2\frac{H}{2n}}{\cosh^2\left(\frac{W}{2n} + j\frac{2k-1}{2n}\pi\right)}. \tag{22}$$

To find p_k from p_k^2 it is convenient to form the two real quantities $X_k = [(p_k^2)(p_k^2)^*]^{1/2}$ and $Y_k = \mathrm{Re}\,(p_k^2)$, from which the p_k are found by

$$p_k = -\sqrt{\frac{X_k + Y_k}{2}} \pm j\sqrt{\frac{X_k - Y_k}{2}} \tag{23}$$

where poles with positive real part are discarded. Thus we find from (22)

$$X_k = \frac{\left[\cosh^2\frac{H}{n} + \cosh^2\frac{W}{n} + \cos^2\frac{2k-1}{n}\pi - 2\cosh\frac{H}{n}\cosh\frac{W}{n}\cos\frac{2k-1}{n}\pi - 1\right]^{1/2}}{\cosh\frac{W}{n} + \cos\frac{2k-1}{n}\pi} \tag{24}$$

$$Y_k = 1 - \frac{\left(1 + \cosh\frac{H}{n}\right)\left(1 + \cosh\frac{W}{n}\cos\frac{2k-1}{n}\pi\right)}{\left(\cosh\frac{W}{n} + \cos\frac{2k-1}{n}\pi\right)^2} \tag{25}$$

where $k = 1, 2, \cdots, (n/2)$ for n even and $k = 1, 2, \cdots, (n+1/2)$ for n odd. The zeros q_k may be found by allowing w to go to ∞. Then X_k and $-Y_k$ become equal to the same value, which we denote as U_k. The zeros are given by $q_k = \pm j\sqrt{U_k}$ where

$$U_k = \frac{\cosh\frac{H}{n} - \cos\frac{2k-1}{n}\pi}{1 + \cos\frac{2k-1}{n}\pi}. \tag{26}$$

We are not primarily interested in finding the p_k and q_k, but rather we wish to find the real polynomials in $\rho(s)$. We therefore write (19) in the form of (16) with Q_n and P_n given by

$$Q_n(s) = \tfrac{1}{2}(1-R)\prod_{k=1}^{[n/2]}(1 + C_k s^2) \tag{27}$$

$$P_n(s) = \tfrac{1}{2}(1+R)\prod_{k=1}^{[n/2]}(1 + B_k s + A_k s^2)\begin{cases}1, & n \text{ even}\\ (1 + A_{(n+1)/2}s), & n \text{ odd}\end{cases} \tag{28}$$

with

$$A_k = 1/X_k \qquad B_k = \sqrt{2(X_k + Y_k)}/X_k \qquad C_k = 1/U_k \tag{29}$$

and $[n/2]$ is $n/2$ for n even and $(n-1)/2$ for n odd. The desired function $z_n(s)$ is found by substituting (27) and (28) into (17).

For the maximally flat transformer we know that

$$|\rho(\phi)|^2 = \frac{1}{1 + (1/w)^2\sec^{2n}\phi} \qquad w = \tfrac{1}{2}|\sqrt{R} - 1/\sqrt{R}|. \tag{30}$$

As before, we write $\rho(s)$ in the form of (16), with Q_n and P_n given by (27) and (28) with $C_k = 0$, and the X_k and Y_k in (29) found from

$$X_k = \sqrt{1 + w^{4/n} - 2w^{2/n}\cos\left(\frac{2k-1}{n}\pi\right)} \tag{31}$$

$$Y_k = 1 - w^{2/n}\cos\left(\frac{2k-1}{n}\pi\right) \tag{32}$$

where $k = 1, 2, \cdots, (n/2)$ for n even, and $k = 1, 2, \cdots, (n+1)/2$ for n odd.

References

[1] R. E. Collin, "Theory and design of wide-band multisection quarter-wave transformers," *Proc. IRE*, vol. 43, pp. 179–185, Feb. 1955.
[2] H. J. Riblet, "General synthesis of quater-wave impedance transformers," *IRE Trans. Microwave Theory Tech.*, vol. MTT-5, pp. 107–114, Apr. 1957.
[3] C. S. Gledhill and A. M. Issa, "Exact solutions of stepped impedance transformers having maximally flat and Chebyshev characteristics," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-17, pp. 379–386, July 1969.
[4] S. B. Cohn, "Optimum design of stepped transmission-line transformers," *IRE Trans. Microwave Theory Tech.*, vol. MTT-3, pp. 16–21, Apr. 1955.
[5] F. C. Chang, H. Y. Yee, and N. F. Audeh, "A graphical method for the design of stepped impedance transformers," *1971 IEEE G-MTT Int. Microwave Symp. Dig.*, IEEE Cat. 71, C-25-M, pp. 4–5, May 1971.
[6] L. Young, "Stepped-impedance transformers and filter prototypes," *IRE Trans. Microwave Theory Tech.*, vol. MTT-10, pp. 339–359, Sept. 1962.
[7] L. Young, "Unit real functions in transmission line circuit theory," *IRE Trans. Circuit Theory*, vol. CT-7, pp. 247–250, Sept. 1960.
[8] P. I. Richards, "Resistor-transmission-line circuit," *Proc. IRE*, vol. 36, pp. 217–220, Feb. 1948.

Dielectric Losses in an H-Plane-Loaded Rectangular Waveguide

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***Abstract*—The attenuation constant due to dielectric losses in a rectangular waveguide loaded with a dielectric slab in the H-plane is calculated. Results for the dominant LSM_{11} mode are presented graphically.**

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